U.G. 4th Semester Examination - 2020

MATHEMATICS

[HONOURS]

Course Code : MTMH-CC-T-8

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

 a) Using the definition of the Darbaux integration, show that the function $f:[0,1]\to R$ defined by f(x)=x is Darbaux integrable.

 b) If *m* and *M* are the infimum and suprimum of a continuous function $f:[-1,1]\to R$, then prove that

$$m\le\frac{1}{2}\int_{-1}^{1}f(x)dx\le M.$$

 c) State the Riemann condition of the Darbaux integrability of a bounded function.

 d) Prove that $\int_{\underline{a}}^{b}f(x)dx\le\int_{a}^{\overline{b}}f(x)dx$.

 e) Show that the function

$$f(x)=\begin{cases}1, & x \text{ is rational number in } [0,1]\\ 0, & x \text{ is irrational number in } [0,1]\end{cases}$$

 is not Riemann integrable over [0,1].

 f) Prove that $\left|\int_{a}^{b}f(x)dx\right|\le\int_{a}^{b}|f(x)|dx$.

 g) Show that the improper integral $\int_{1}^{0}\frac{dx}{\sqrt{1-x^2}}$ is convergent.

 h) Let $f(x)=\begin{cases}\frac{1}{x}, & x\ne 0\\ 0, & x=0.\end{cases}$

 Find the Cauchy principal value of the improper integral $\int_{-1}^{1}f(x)dx$.

 i) Show that the sequence of functions $\left\{\frac{nx}{1+nx}\right\}, x\in[0,1]$ is point-wise convergent on [0,1]. Find its limit function.

 j) Give an example of a sequence of functions which point-wise converges to a function, but not uniformly converges to that function.

k) Show that the series

$\sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$ is uniformly convergent on entire real line *R*.

l) Find the radius of convergence of the power series

$$\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \ldots$$

m) If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have the same radius of convergence R>0 and have the same limit function $f(x)$ in $(-R, R)$. Then show that two series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are identical.

n) State the *Riemann–Lebesgue Lemma* relate to Fourier series.

o) Let $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier series of the function $f(x) = x, -2 \le x \le 2$. Then find b_n for all $n \in N$.

2. Answer any **four** questions: 5×4=20

a) If $f:[a,b] \to R$ be a Riemann intergrable function, then the function is bounded. Give an example to show that converse of the above result is not true.

b) Define a piecewise continuous function. Prove or disprove that every bounded piecewise continuous function $f:[a,b] \to R$ is Riemann integrable.

c) Define Gamma functions $\Gamma(x)$ as an improper integral. Show that it is convergent if $x > 0$.

d) State and prove the Cauchy criterion for uniform convergence of series of functions.

e) Prove that the uniform limit of a sequence of continuous functions is continuous.

f) Let

$$f(x) \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ 1 & \text{for } 0 < x < \pi \\ 0 & \text{for } x = \pi \end{cases}$$

Find the Fourier coefficients of the function f.

g) Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1 and the series of real numbers $\sum_{n=0}^{\infty} a_n$ converges to a. Then show that $\lim_{x\to 1-0} f(x)=f(1)=a$.

3. Answer any **two** questions from **(a)** to **(d)**: 10×2=20

a) i) Let $f:[a,b]\to[m,M]$ be a Riemann integrable function over [a, b] and $g:[m,M]\to R$ be a continuous function. Then prove that the composition function $g\circ f:[a,b]\to R$ is Riemann integrable over [a, b].

ii) State and prove fundamental theorem of integral calculus. Apply the theorem to evaluate $\int_1^2 F(x)dx$, where $F(x)=3x^2-1$ for $x\in[1,2]$. 5+(4+1)

b) i) For a bounded function $f:[a,b]\to R$, show that the following conditions are equivalent:

I) $f:[a,b]\to R$ is Riemann integrable over [a,b] with $\int_a^b f(x)dx = A$.

II) $\int_{\underline{a}}^{b} f(x)dx=\int_a^{\overline{b}} f(x)dx = A$.

III) Let $P=\{x_0,x_1,...,x_r,x_{r-1},...,x_n\}$, $t_r\in[x_r,x_{r-1}]$ for r = 1, 2..., n and $\delta=\max\{x_r-x_{r-1}:i=1,2,...n\}$ and $S(P,f)=\sum_{r=1}^{n} f(t_r)(x_r-x_{r-1})$.

Then $\lim_{\delta\to 0} S(P,f)=A$.

ii) If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence a. Then find the radius of convergence of the derived series $\sum_{n=0}^{\infty} b_n x^n$ obtained from $\sum_{n=0}^{\infty} a_n x^n$ by term-by term differentiations. 7+3

c) i) Let $\{f_n\}$ be a sequence of differentiable functions defined on [a, b] such that

I. $\{f_n(x_0)\}$ converges for some $x_0\in[a,b]$.

II. $\{f_n'\}$ converges uniformly on [a, b].

Then $\{f_n\}$ converges uniformly to some function f on [a, b] and $f'(x)=\lim_{n\to\infty} f_n'(x)$.

ii) State and prove Weierstrass M-test for series of functions. 6+4

d) i) Establish the Bessel's *inequality* related to Fourier series.

ii) From the equality

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}, -\pi \le x \le \pi,$$

prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \text{ and } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$ 6+4

———————